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Transient state temperature distribution in a cylindrical conductor with skin effect

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1. INTRODUCTION

IN A PREVIOUS paper [1] we examined the problem of the temperature distribution in a cylindrical conductor in the steady-state regime and in the presence of skin effect. We solved a set of couple equations involving the space distribution of temperature $T_p(r)$ and J_m which represents the maximum current density. The latter is allowed to be a complex quantity. We assumed that all physical parameters such as the thermal conductivity λ and the electrical resistivity ρ are constants. After solving the equation involving J_m by a method which is described in detail in ref. [1], we obtained the following equation governing the distribution of temperature in the presence of skin effect:

$$\frac{d^2T_p}{dr^2}+\frac{1}{r}\frac{dT_p}{dr}+\Gamma K(r/R)=0 \tag{1}$$

where Γ and $K(r/R)$ are defined by

$$\Gamma=\frac{\rho}{2\lambda}\frac{I_m^2p}{4\pi^2r_0^2}\frac{1}{(\mathrm{ber}'\sqrt{p}r_0)^2+(\mathrm{bei}'\sqrt{p}r_0)^2} \tag{2}$$

$$K(r/R)=1+\sum_{n=1}^{\infty}\frac{1}{(n!)^3(2n-1)!!}\left(\frac{r}{R}\right)^{4n} \tag{3}$$

r_0 is the radius of the conductor and $p=|\xi|=2\pi f\mu\gamma$, $R=2^{5/4}/p^{1/2}$, f is the current frequency, μ the magnetic permeability and $\gamma=1/\rho$. We note that here, $T_p(r)$ represents the absolute temperature at a distance r from the centre line of the conductor in the steady-state regime. The solution of equation (1) satisfying the classical boundary conditions [1] was found as

$$T_p(r)=\Gamma\left[r_0^2A(r/R)+\frac{\lambda r_0}{\varepsilon}B(r_0/R)-r^2A(r/R)\right] \tag{4}$$

where the analytical expressions of the functions $A(r/R)$ and $B(r/R)$ are given in ref. [1]. Here we choose to illustrate them graphically as a function of r together with $K(r/R)$ in Fig. 1. The aim of this note is to extend this problem to the transient state distribution of temperature in the presence of skin effect.

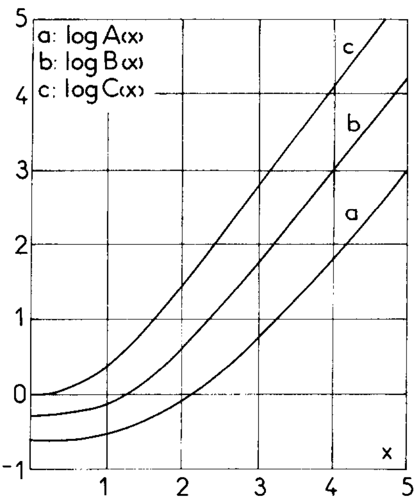

FIG. 1. Curves representing the variation with x of the functions: (a) log $A(x)$ (see equation (25) of ref. [1]); (b) log $B(x)$ (see equation (26) of ref. [1]); (c) log $K(x)$ (see equation (3) of this work).

2. TRANSIENT STATE

The equation governing the temperature distribution in space and time for a cylindrical conductor in the presence of skin effect is

$$\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\Gamma K(r/R)=\frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{5}$$

where $T(r,t)$ is written as

$$T(r,t)=T_p(r)+T_t(r,t). \tag{6}$$

$T_t(r,t)$ represents the transient state component of the temperature and $\kappa=\lambda/(c\delta)$ is the thermal diffusivity, c the specific heat and δ the mass density. To solve this equation one needs, besides the boundary conditions specified in ref. [1], an initial condition which is chosen as

$$T(r,t=0)=0 \quad \text{or} \quad T_t(r,t=0)=-T_p(r), \quad 0\leqslant r\leqslant r_0. \tag{7}$$

Substituting equation (6) into equation (5), one obtains a set of two equations. The first one is identical to equation (1), and hence, gives $T_p(r)$, and the second one involves $T_t(r,t)$

$$\frac{\partial^2T_t}{\partial r^2}+\frac{1}{r}\frac{\partial T_t}{\partial r}=\frac{1}{\kappa}\frac{\partial T_t}{\partial t}. \tag{8}$$

This equation is solved by the standard method of separation of variables [2]. One can easily verify that the result takes the form

$$T_t(r,t)=-\sum_{n=1}^{\infty}A_nJ_0\left(\frac{\alpha_n}{r_0}r\right)\mathrm{e}^{-(\alpha_n^2/r_0^2)\kappa t}. \tag{9}$$

This result shows immediately that the steady-state temperature $T_p(r)$ can be expanded in a series of Bessel functions of zeroth order

$$-T_t(r,t=0)=T_p(r)=\sum_{n=1}^{\infty}A_nJ_0\left(\frac{\alpha_n}{r_0}r\right). \tag{10}$$

To determine the quantity A_n, we need to use both equations (4) and (10). This leads to

$$A_n=2\frac{T_{p,\max}r_0^2J_1(\alpha_n)-\Gamma\alpha_n\int_0^{r_0}r^3A(r/R)J_0\left(\frac{\alpha_n}{r_0}r\right)\mathrm{d}r}{r_0^2\alpha_n[J_0^2(\alpha_n)-J_1^2(\alpha_n)]} \tag{11}$$

where $T_{p,\max}$ is the maximum value of the temperature which is reached at the centre line, $J_1(\alpha_n)$ the Bessel function of first order, α_n the solution of the equation

$$J_0(\alpha) - \frac{\lambda}{r_0 \varepsilon}\alpha J_1(\alpha) = 0 \qquad (12)$$

and ε represents the convective heat transfer coefficient. Substituting equations (9) and (10) into equation (6) yields

$$T(r,t) = \sum_{n=1}^{\infty} A_n(1 - e^{-(\alpha_n^2/r_0^2)\kappa t}) J_0\left(\frac{\alpha_n}{r_0} r\right). \qquad (13)$$

To illustrate the importance of skin effect on the transient state temperature distribution, we have plotted in Fig. 2 the variation of $\Delta T(r,t) = T(r,t) - T_s(t)$ as a function of r for various times from the moment where the high frequency current is switched on. $T_s(t)$ represents the absolute temperature at the surface of the conductor. To plot these curves, we have used the same values for the physical parameters λ, ρ, r_0 and I_m as in ref. [1]. I_m is chosen to represent the case of a strong skin effect corresponding to a high current frequency ($f = 0.9$ MHz). Furthermore, the diffusivity κ is taken as

$$\kappa = 21.3 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$$

which corresponds to tungsten at 3000 K [3]. One observes that the skin effect develops very quickly after switching the current on. Indeed, at short times, one immediately sees a peak of temperature near the surface, the maximum of which is located around the skin depth $\delta = (\pi f \mu \gamma)^{-1/2} \approx 0.5$ mm. It is clear that for short times ΔT should be negative since at $t = 0$, we have used the initial condition $T(r, t = 0) = 0$. As time increases, the transient state temperature distribution evolves progressively towards the steady-state distribution which, in our case is reached after about 30 s.

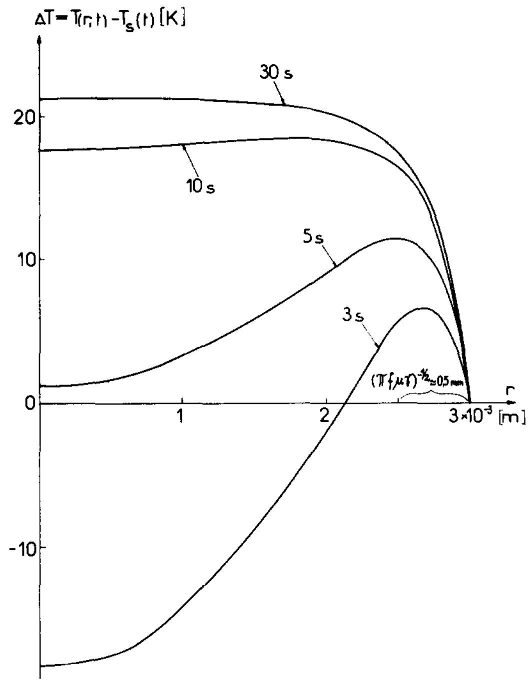

FIG. 2. The variation of $\Delta T(r,t) = T(r,t) - T_s(t)$ as a function of r at various times: $t = 3$ s, $T_s = 1925.72$ K; $t = 5$ s, $T_s = 2454.57$ K; $t = 10$ s, $T_s = 2901.52$ K; $t = 30$ s, $T_s = 3002.76$ K (steady-state temperature distribution).

3. CONCLUSION

In this note, we present an exact analytical solution for the transient state temperature distribution in an electrical conductor in the case of a strong skin effect. This solution can be exploited in various practical applications such as induction heating systems, determination of physical parameters for a given temperature profile, etc. We have deliberately chosen the case of a strong skin effect to illustrate clearly its impact on the temperature distribution. But this formalism can be of course applied to the limit of a weak skin effect as well. The physical parameters involved in this problem are all assumed to be constant which leads to a linear system to be solved. In the case where these parameters are functions of temperature, one is faced with a non-linear system which is far more difficult to solve. We hope to examine this problem in the near future using the optimal linearization method which we have recently developed [4].

REFERENCES

1. A. Jordan, A. Szybiak, M. Benmouna and A. Barka, Temperature distribution in a cylindrical conductor with skin effect, *Int. J. Heat Mass Transfer* **30**, 1539–1541 (1987).
2. A. B. De Vriendt, *La Transmission de la Chaleur* (Edited by Gaëtan Morin), Vol. 1, Tome 1. Quebec, Canada (1982).
3. R. E. Taylor, Thermal properties of tungsten SRM's 730 and 799, *J. Heat Transfer* **100**, 330–333 (1978).
4. A. Jordan, S. Khaldi, M. Benmouna and A. Borucki, Study of non-linear heat transfer problems, *Revue Phys. Appl.* **22** (1987).